Solution of Heat equation using Adomian Decomposition Method

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Abstract : In this paper, the Adomian Decomposition Method (ADM) is applied to some parabolic (heat) equations subjected to the initial and non-local boundary conditions to obtain the approximate solution. The method shows an accurate and efficient technique in comparison with existing exact solutions.

Keywords : Adomian decomposition method, heat equations, exact solution, non-local boundary conditions, series solution.

1 Introduction

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Adomian decomposition method (ADM) was introduced by Adomian G(1923-1996) and is one of a semi-analytical method for solving approximate (series) solutions for large classes of non- linear and linear differential equations. The classical ADM has been reported to be a very good method for solving strong non-linear mathematical models. Since then, this method has been known as Adomian Decomposition Method [1, 2]. The main aim of the is to provide a simple and easy method to handle a heat equation by using ADM in the frame of a symbolic computer program so that by giving linear operator, it generates initial guess, integral inverse of the linear operator, recursive relation and the terms of series solution automatically. Consider the application to the heat equation in the form

$$\nabla^2 u - \frac{\partial u}{\partial t} = g(x,y,z,t) \tag{1.1}$$

The heat equation is considered because it is an equation of physical significance; it obviously is solvable by other methods so does not by itself show the power of the method unless we consider non-linear and/or stochastic modifications[3]. It merely illustrates a new and valuable manner of solution with a familiar problem as well as providing an introduction to a method which can now very complicated systems[2]. Cheniguel A and Ayadi A[4] have presented for solving heat equation with an initial condition and non-local boundary conditions using ADM. The numerical applications discussed here, show that the obtained solution coincides with the exact one. Ali E. J[5] has presented a reliable framework by applying a new technique for treatment to initial boundary value problems by mixed initial and boundary conditions together to obtain a new initial solution at every iterations using Adomian decomposition method. Catal S[6] has discussed the solutions of linear and non-linear partial differential equations by Differential transform method (DTM) and Adomian decomposition method (ADM). In this work, the ADM is adopted to solve some numerical examples on one dimensional linear parabolic(heat) equation with non-local boundary conditions which yields the exact solution of problem.

2. Adomian Decomposition method

In this section, we outline the steps to obtain a solution of There has recently been a lot of attention to the search for better and more accurate solution methods for determining approximate or exact solution to one dimensional heat equation with non-local boundary conditions.

Consider the one dimensional heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2} + f(x,t), \qquad x \in (0,1),\ t \in (0,T) \qquad (2.1)$$

subject to the initial condition:

$$u(x,0) = g(x), \qquad x \in [0,1] \qquad (2.2)$$

and the non-local boundary conditions:

$$u(0,t) = \int_0^1 \phi(x,t)u(x,t)dx + g_1(t), \quad t \in (0,T] \ (2.3)$$

$$u(1,t) = \int_0^1 \psi(x,t)u(x,t)dx + g_2(t), \ \ t \in (0,T] \ \ (2.4)$$

where $f, g, g_1, g_2, \phi, \psi$ sufficiently smooth known functions and T is given constant .A number of authors as have suggested traditional techniques for solving this type of problems. In this work, we present a technique as suggested by Cheniguel A. and Ayadi A. [4] based on Adomian series solution method which yields the exact solution of problem (2.1) to (2.4).

For the purpose of the solution of these equations it is convenient to rewrite (2.1) in the operator form.

$$L_t(u) = L_{xx}(u) + f(x,t) \qquad (2.5)$$

where the differential operators are defined as :

$$L_t(.) = \frac{\partial}{\partial t}(.) \ \ and \ \ L_{xx} = \frac{\partial^2}{\partial x^2}(.)$$

And the inverse operator L_t^{-1}, provided that it exists, is defined as :

$$L_t^{-1} = \int_0^t (.)\,dt \qquad (2.6)$$

Applying the inverse operator on both sides of (2.5) and using the initial condition, obtain:

$$L_t^{-1}(L_t(u)) = L_t^{-1}(L_{xx}(u)) + L_t^{-1}(f(x,t)) \quad (2.7)$$

Simplifying (2.7), we obtain:

$$u(x,t) = g(x) + L_t^{-1}(L_{xx}(u)) + L_t^{-1}(f(x,t)) \quad (2.8)$$

The ADM consists of decomposing the unknown function $u(x,t)$ as sum of an infinite number of components defined by the decomposition series.

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \qquad (2.9)$$

where u_0 is defined as $u(x,0)$. The components $u_n(x,t)$ are obtained by the recursive formula:

$$\sum_{n=0}^{\infty} u_n(x,t) = g(x) + L_t^{-1}(L_{xx}(\sum_{n=0}^{\infty} u_n(x,t))) + L_t^{-1}(f(x,t)),\ n \geq 0 \qquad (2.10)$$

$$u_0(x,t) = g(x) + L_t^{-1}(f(x,t)) \qquad (2.11)$$

$$u_{n+1}(x,t) = L_t^{-1}(L_{xx}(u_n(x,t))),\ n \geq 0 \qquad (2.12)$$

We note that the recursive relationship is constructed on the basis that the component $u_0(x,t)$ is defined by all terms that arise from the initial condition and from integrating the source term.

According, considering the first few terms, an equation (2.8) and (2.9) gives:

$$\begin{aligned} u_0 &= g(x) + L_t^{-1}(f(x,t)) \\ u_1 &= L_t^{-1}(L_{xx}(u_0)) \\ u_2 &= L_t^{-1}(L_{xx}(u_1)) \quad \text{and so on} \end{aligned} \qquad (2.13)$$

In view of (2.13), the components $u_0, u_1, u_2, u_3, \ldots\ldots\ldots u_n$ are completely determined. As a result $u(x,t)$ of the heat equation (2.1) is readily obtained in a series form by using the series as assumption in (2.9).

3. Numerical examples

Example 1: Consider the heat equation:

$$u_t = u_{xx}$$

Initial conditions: $u(x,0) = \sin(\pi x)$

Exact solution: $u(x,t)=\sin(\pi x)e^{-\pi^2 t}$

Solution:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+f(x,t)$$

$$f(x,t)=0$$

$$\therefore \frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$$

$$u(x,0)=f(x)=\sin(\pi x)$$

We write this problem in an operator form and apply the above developments and using (2.11), to obtain solution as:

$$u_0=u(x,0)+L_t^{-1}(f(x,t))=\sin\pi x$$

$$u_1=L_t^{-1}(L_{xx}(u_0))=-\pi^2 t\sin\pi x$$

$$u_2=L_t^{-1}(L_{xx}(u_1))=(-\pi^2 t)^2\frac{\sin\pi x}{2!}$$

$$u_3=L_t^{-1}(L_{xx}(u_2))=(-\pi^2 t)^3\frac{\sin\pi x}{3!}$$ and so on.

Applying Eq(2.9), the solution in the series form is given by

$$u(x,t)=\sin\pi x\, e^{-\pi^2 t}$$

Example 2: Consider the heat equation:

$$g(x)=\cos\left(\frac{\pi}{2}x\right) \qquad 0<x<1$$

$$g_1(t)=\exp\left(\frac{-\pi^2}{4}t\right), \qquad 0<t<1$$

$$g_2(t)=\frac{-2}{\pi}\exp\left(\frac{-\pi^2}{4}t\right), \qquad 0<t<1$$

Exact solution: $u(x,t)=\exp\left(\frac{-\pi^2}{4}t\right)\cos\left(\frac{\pi}{2}x\right)$

Solution:

$$g(x)=\cos\left(\frac{\pi}{2}x\right) \quad ; \quad f(x,t)=0$$

We write this problem in an operator form and apply the above developments and using (2.11), to obtain solution as:

$$u_0(x,t)=\cos\left(\frac{\pi}{2}x\right)$$

$$u_1==\frac{-\pi^2 t}{4}\cos\left(\frac{\pi}{2}x\right), \quad u_2=\left(\frac{-\pi^2 t}{4}\right)^2\frac{1}{2!}\cos\left(\frac{\pi}{2}x\right)$$

$$u_3=\left(\frac{-\pi^2 t}{4}\right)^3\frac{1}{3!}\cos\left(\frac{\pi}{2}x\right)$$ and so on

Applying Eq.(2.9), the solution in the series form is given by

$$u(x,t)=\cos\left(\frac{\pi}{2}x\right)\left[\left(1-\left(\frac{-\pi^2 t}{4}\right)+\left(\frac{-\pi^2 t}{4}\right)^2\frac{1}{2!}-\left(\frac{-\pi^2 t}{4}\right)^3\frac{1}{3!}+\ldots\ldots\ldots\ldots\right)\right]$$

$$u(x,t)=\cos\left(\frac{\pi}{2}x\right)\exp\left(\frac{-\pi^2 t}{4}\right)$$

Example 3: Consider the heat equation:

$$u_t=\frac{1}{4}u_{xx} \qquad 0\le x\le 2$$

Initial conditions :

$$u(x,0)=2\sin\left(\frac{\pi x}{2}\right)-\sin(\pi x)+4\sin(2\pi x)=f(x)$$

Boundary conditions

$$u(0,t)=u(2,t)=0$$

Exact solution:

$$u(x,t)=2\sin(\pi x/2)\exp(-\pi^2 t/16)-\sin(\pi x)\exp(-\pi^2 t/4)+4\sin(2\pi x)\exp(-\pi^2 t)$$

Solution **:** We write this problem in an operator form and apply the above developments and using (2.11), to obtain solution as:

$$u_0 = 2\sin\left(\frac{\pi x}{2}\right) - \sin\left(\pi x\right) + 4\sin\left(2\pi x\right)$$

$$u_1 = \frac{1}{4}\left[\left(\frac{-\pi^2 t}{4}\right)2\sin\frac{\pi x}{2} + \pi^2 t\sin\left(\pi x\right) - \left(4\pi^2 t\right)4\sin\left(2\pi x\right)\right]$$

$$u_2 = \frac{1}{16}\left[\left(\frac{\pi^4 t^2}{32}\right)2\sin\frac{\pi x}{2} - \frac{\left(\pi^2 t\right)^2}{2}\sin\left(\pi x\right) + \frac{\left(4\pi^2 t\right)^2}{2}4\sin\left(2\pi x\right)\right]$$

and so on.

Applying Eq.(2.9), the solution in the series form is given by

$$u(x,t) = 2\sin(\pi x/2)\exp(-\pi^2 t/16) - \sin(\pi x)\exp(-\pi^2 t/4) + 4\sin(2\pi x)\exp(-\pi^2 t)$$

4. Conclusion

In this paper, the ADM has been applied to heat equations subjected to initial and non-local boundary conditions. In order to obtain very accurate solutions, the domain(region) has been splitted into subintervals and the approximating solutions are obtained. The ADM develops from the difference equation system with initial conditions, a recurrence equation system that finally leads to the series solution. The results obtained show that the ADM gives the exact solution which shows that these approximations are so compatible with real ones.

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